International Journal of Theoretical Physics, Vol. 37, No. 1, 1998

The Nondemolition Measurement of Quantum Time

V. P. Belavkin[1] and M. G. Perkins[1]

Received July 4, 1997

The problem of time operator in quantum mechanics is revisited. The unsharp measurement model for quantum time based on the dynamical system–clock interaction is studied. Our analysis shows that the problem of the quantum time operator with continuous spectrum cannot be separated from the measurement problem for quantum time.

1. INTRODUCTION: THE TIME OPERATOR

The problem of time measurement in quantum theory cannot be solved within the von Neumann (1955) theory simply by defining the corresponding self-adjoint operator as a generator of the shift for the energy of a physical system S (assumed to have a positive spectrum, $\varepsilon \in \mathbb{R}_+$), as no such operator exists in the Hilbert space $\mathcal{H}_S$.

For the purpose of simplicity let us study this problem for a quantum system with a continuous (unbounded) energy spectrum of constant degeneracy; for the case of a free quantum particle see Holevo (1982). The system Hilbert space can be decomposed into a family of eigenspaces $\mathcal{H}_\varepsilon$ ($\varepsilon \in \mathbb{R}_1$) of fixed energy. The dimensionality of $\mathcal{H}_\varepsilon$ corresponds to the degeneracy of the eigenvectors corresponding to ε. We represent the state vectors $\psi \in \mathcal{H}_S$ by a family $\{\psi(\varepsilon)|\varepsilon \geq 0\}$ of Hilbert space vectors $\psi(\varepsilon) \in \mathcal{H}_\varepsilon$ such that $\int_0^\infty \|\psi(\varepsilon)\|^2 \, d\varepsilon = 1$.

Now, without loss of generality, we can treat all $\psi(\varepsilon)$ as elements of some Hilbert space $\mathcal{H}$. This is because all the $\mathcal{H}_\varepsilon$ can each be embedded in the same $\mathcal{H}$, so for all ε, $\psi(\varepsilon) \in \mathcal{H}_\varepsilon \subseteq \mathcal{H}$. Then we can describe each state

[1] Mathematics Department, University of Nottingham, England NG7 2RD; e-mail address: vp@@maths.nott.ac.uk.

0020-7748/98/0100-021 9$15.00/0

vector $|\psi\rangle$ by an analytic [on the upper half-plane, where $\mathrm{Im}(\tau) > 0$] function $h:\mathbb{C} \mapsto \mathcal{H}$:

$$h(\tau) = \frac{1}{\sqrt{2\pi\hbar}} \int_0^\infty e^{i\varepsilon\tau/\hbar}\psi(\varepsilon)\, d\varepsilon$$

which is completely defined by its value on $\mathbb{R}$. This analytic representation restricted to $\tau \in \mathbb{R}$ is called the time representation. The Hilbert space $\tilde{\mathcal{H}}_S \simeq \mathcal{H}_S$ of these analytic functions with the squared norm $\|h\|^2 := \int_{-\infty}^{\infty}\|h(\tau)\|^2 d\tau = \|\psi\|^2$ is embedded as one half into the Hilbert space $L^2_{\mathcal{H}}(\mathbb{R})$ of all square-integrable functions of $\mathbb{R}$ with values in $\mathcal{H}$.

In this enlarged space $L^2_{\mathcal{H}}(\mathbb{R})$ there exists a self-adjoint operator $\hat{\tau}$ defined by the multiplication $[\hat{\tau}h]\,(\tau) = \tau h(\tau)$ with the eigen-spectral family $\{E_t\!: t \in \mathbb{R}\}$ of orthoprojectors $[E_t h]\,(\tau) = 1_t(\tau)h(\tau)$, where $1_t(\tau) = 0$ for $t \le \tau$, and $= 1$ for $t > \tau$.

However, the operator $\hat{\tau}$ does not leave the physical subspace $\tilde{\mathcal{H}}_S \subset L^2_{\mathcal{H}}(\mathbb{R})$ invariant. Instead, the unitary operator $U_\lambda = e^{i\lambda\tau/\hbar}$ functions as an isometry $h(\tau) \mapsto e^{i\lambda\tau/\hbar}h\,(\tau)$ on $\mathcal{H}_S$ corresponding to the shift $|\varepsilon\rangle \mapsto |\varepsilon + \lambda\rangle$ on $\mathcal{H}_S$ for each $\lambda > 0$. Note that the isometry on $\tilde{\mathcal{H}}_S$ is adjoint not to U_λ^{-1}, but to the energy shift operator V_λ in $\tilde{\mathcal{H}}_S$ given by $[V_\lambda\psi]\,(\varepsilon) = \psi(\varepsilon + \lambda)$. The operator $V_\lambda^\dagger$ in this representation acts as

$$[V_\lambda^\dagger\psi](\varepsilon) := \begin{cases} \psi(\varepsilon - \lambda), & \varepsilon > \lambda \\ 0, & \varepsilon \le \lambda \end{cases}$$

Indeed, if $\psi(\varepsilon) = (2\pi\hbar)^{-1/2} \int_{-\infty}^{\infty} e^{-i\varepsilon\tau/\hbar}h(\tau)d\tau = 0$ for all $\varepsilon < 0$, then

$$\sqrt{2\pi\hbar}[e^{i\lambda\hat{\tau}/\hbar}h](\tau) = \int_0^\infty e^{i(\varepsilon+\lambda)\tau/\hbar}\psi(\varepsilon)\, d\varepsilon = \int_0^\infty e^{i\varepsilon\tau/\hbar}[V_\lambda^\dagger\psi](\varepsilon)d\varepsilon$$

i.e., $e^{i\lambda\hat{\tau}/\hbar}h$ is analytic in the upper half-plane, so the operator U_λ leaves $\tilde{\mathcal{H}}_S$ invariant. Note that the shift operator $V_\lambda = \mathrm{P}_0 U_\lambda^{-1}$ given by the orthoprojector P_0 in $L^2_{\mathcal{H}}(\mathbb{R})$ onto $\tilde{\mathcal{H}}_S = L^2_{\mathcal{H}}(\mathbb{R}_+)$ is defined on $\mathcal{H}_S$ by

$$V_\lambda|\varepsilon\rangle = \begin{cases} |\varepsilon - \lambda\rangle, & \lambda \le \varepsilon \\ 0, & \lambda > \varepsilon \end{cases} \qquad [\mathrm{P}_\lambda\psi](\varepsilon) = \begin{cases} \psi(\varepsilon), & \varepsilon > \lambda \\ 0, & \varepsilon \le \lambda \end{cases}$$

Here $\mathrm{P}_\lambda = V_\lambda^\dagger V_\lambda$ is the orthoprojector, giving the kernel $I - \mathrm{P}_\lambda$ for V_λ. So the operator V_λ is not isometric, but only coisometric in $\mathcal{H}_S$ as $[V_\lambda\psi](\varepsilon) = 0$ for all ε if ψ is localized as $\psi(\varepsilon) = 0$ for $\varepsilon \ge \lambda$.

Although the operators V_λ are not normal and do not commute, they have the overcomplete analytic family $\{|s)\ |\mathrm{Re}(s) > 0\}$ of nonorthonormal common eigenvectors, given by the Laplace transform $|s) = \int_0^\infty e^{-\varepsilon s}|\varepsilon\rangle \, d\varepsilon$ of the generalized basis $\{|\varepsilon\rangle | \varepsilon \in \mathbb{R}_+\}$. The proof is as follows:

$$V_\lambda|s) = \int_0^\infty e^{-\varepsilon s} V_\lambda |\varepsilon\rangle \, d\varepsilon = \int_\lambda^\infty e^{-\varepsilon s}|\varepsilon - \lambda\rangle \, d\varepsilon = e^{-\lambda s}|s)$$

Let us show that the vectors $|s)$ labeled by complex numbers s are not orthogonal, and are normalizable only if $\mathrm{Re}(s) > 0$. Indeed,

$$(s|s') = \int_0^\infty \int_0^\infty e^{-(\varepsilon'\bar{s}+\varepsilon s')} \langle \varepsilon' | \varepsilon \rangle \, d\varepsilon \, d\varepsilon' = \int_0^\infty e^{-(\bar{s}+s')} \, d\varepsilon$$

since $\langle \varepsilon' | \varepsilon \rangle = \delta(\varepsilon - \varepsilon')$. But $\int_0^\infty e^{-\varepsilon(\bar{s} + s')} \, d\varepsilon = 1 / (\bar{s} + s')$, and so the vectors are not orthogonal. If $s = s'$, then

$$(s|s) = \frac{1}{2 \, \mathrm{Re}(s)} = \frac{1}{2k} < \infty \qquad \text{if} \quad k > 0$$

where $s = k + i\hbar^{-1}\tau$.

This family is complete in $L^2(\mathbb{R}_+)$ and hence in $\mathcal{H}_S$ in the sense that every state vector $\psi \in \mathcal{H}_S$ can be written as an integral span

$$\psi = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} |s)\eta(s^*) \, ds$$

along any path from $-i\infty$ to $i\infty$ in the domain of analyticity of the function $\eta(s^*)$, where $\eta(s) = (s|\psi$ and $s^* = -\bar{s}$. The completeness relation, written for each component $\psi(\varepsilon) = \langle \varepsilon | \psi$, is simply the inversion of the Laplace *-transform

$$\eta(s^*) = \int_0^\infty (s^*|\varepsilon\rangle \psi(\varepsilon) \, d\varepsilon, \qquad (s^*|\varepsilon\rangle = e^{s\varepsilon}$$

since $\psi(\varepsilon) = (2\pi i)^{-1} \int_{-i\infty}^{i\infty} \langle \varepsilon | s) \, \eta \, (s^*) ds$, $\langle \varepsilon | s) = e^{-s\varepsilon}$. This means that the vector functions $\eta \, (k + i\hbar^{-1} \tau) = \sqrt{2\pi\hbar} h(\tau + i\hbar k)$ define a representation of state vectors $\psi \in \mathcal{H}_S$ in the space of $*$-analytic functions $\eta(s)$ with the inner product

$$\langle \eta' | \eta \rangle = \frac{1}{(2\pi)^2} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} \frac{1}{\bar{s} + s'} \langle \eta'(\bar{s}) | \eta(\bar{s}') \rangle \, d\bar{s} \, ds'$$

given by the kernel $2\pi/(\bar{s} + s')$, as it coincides with

$$\langle \psi' | \psi \rangle = \int_0^\infty \langle \psi' \, (\varepsilon) | \psi \, (\varepsilon) \rangle \, d\varepsilon.$$

However, this inner product can also be expressed as the single integral

$$\langle h'|h\rangle = \lim_{k\to 0}\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\langle\eta'(k+i\hbar^{-1}\tau)|\eta(k+i\hbar^{-1}\tau)\rangle\, d\tau$$

Indeed,

$$\int_{-\infty}^{\infty}\|\eta(k+i\hbar^{-1}\tau)\|^2\, d\tau$$
$$= \int_{-\infty}^{\infty}\left(\int_0^{\infty}\int_0^{\infty} e^{-k(\varepsilon+\varepsilon')+i\tau(\varepsilon-\varepsilon')/\hbar}\langle\psi(\varepsilon')|\psi(\varepsilon)\rangle\, d\varepsilon\, d\varepsilon\right) d\tau$$

Now, since $\int_{-\infty}^{\infty} e^{ix\tau/\hbar}d\tau = 2\pi\hbar\delta(x)$, we obtain

$$\int_{-\infty}^{\infty}\|\eta(k+i\hbar^{-1}\tau)\|^2\, d\tau = 2\pi\hbar\int_0^{\infty} e^{-2k\varepsilon}\|\psi(\varepsilon)\|^2\, d\varepsilon$$

Given that the family of vectors $|s)$ is nonorthogonal, this means that it is over complete. The equality is true for all ψ and since $\psi(\varepsilon) = \langle\varepsilon|\psi$ and

$$\eta\,(k+i\hbar^{-1}\tau) = (k+i\hbar^{-1}\tau|\psi,$$

then this can be written equivalently as

$$\int_{-\infty}^{\infty}|k+i\hbar^{-1}\tau)(k+i\hbar^{-1}\tau|\, d\tau = 2\pi\hbar\int_0^{\infty} e^{-2k\varepsilon}|\varepsilon\rangle\langle\varepsilon|d\varepsilon = 2\pi\hbar e^{-2kH}$$

where H is the induced Hamiltonian of the system in $L^2(\mathbb{R}_+)$. In the limit as $k\to 0$, we obtain

$$2\pi\hbar\|h\|^2 = 2\pi\hbar\int_{-\infty}^{\infty}\|h(\tau)\|^2\, d\tau = 2\pi\hbar\int_0^{\infty}\langle\psi(\varepsilon)|\psi(\varepsilon)\rangle\, d\varepsilon = 2\pi\hbar\|\psi\|^2$$

that is, $\int_{-\infty}^{\infty}|i\hbar^{-1}\tau)\,(i\hbar^{-1}\tau|\, d\tau = 2\pi\hbar\hat{1}$.

2. THE IDEAL UNSHARP MEASUREMENT OF TIME

Let us consider the nonorthonormal family of right eigenvectors $\{|s): \mathrm{Re}(s) = 0\}$ for the coshift operators V_λ at the limit $\mathrm{Re}(s)\to 0$. Now

$$\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}|i\hbar^{-1}\tau)(i\hbar^{-1}\tau|\, d\tau = \frac{1}{2\pi\hbar}\lim_{k\to 0}\int_{-\infty}^{\infty}|k+i\hbar^{-1}\tau)(k+i\hbar^{-1}\tau|d\tau = \hat{1}$$

and we have the normalization condition $\int\|h(\tau)\|^2\, d\tau = 1$ if $\|\psi\|^2 = 1$. So we can treat

$$h(\tau) = \frac{1}{\sqrt{2\pi\hbar}} (i\hbar^{-1}\tau|\psi = \frac{1}{\sqrt{2\pi\hbar}} \eta(i\hbar^{-1}\tau)$$

as the probability amplitude of a time measurement (τ-measurement) described by the continuous overcomplete family of generalized vectors

$$\chi(\tau) = \frac{1}{\sqrt{2\pi\hbar}} |i\hbar^{-1}\tau), \qquad \tau \in \mathbb{R}$$

For each Borel subset Δ, the integral $\int_\Delta |i\hbar^{-1}\tau)\,(i\hbar^{-1}\tau| d\tau$ defines the unsharp, positive, contractive operator Π_Δ acting as

$$\Pi_\Delta\psi = \int_\Delta \chi(\tau) h(\tau)\, d\tau$$

with $h(\tau) = \chi(\tau)^\dagger\psi$. The map $\Delta \mapsto \Pi_\Delta$ defines a positive operator-valued measure, normalized to the identity operator: $\Pi_{\mathbb{R}+} = \hat{1}$ in $\tilde{\mathcal{H}}_S$ and so in $\mathcal{H}_S$. However, this measure is not orthogonal (projector-valued) and this is why it describes the unsharp (fuzzy) measurement of the time in initial state ψ. This measurement gives the best results among all the unsharp measurements of the time parameter of a coherent quantum signal under the maximal likelihood criterion (Helstrom, 1976; Holevo, 1978). However, the nonorthogonal vectors $\chi(\tau)$ cannot be regarded as the time eigenstates that are not yet normalized, as they are not normalizable. Moreover, such ideal measurements demolish the quantum system because there is no way to obtain the *a posteriori* state vectors $\psi_\tau \in \mathcal{H}_S$, compatible with this measurement, using the projection or any other reduction postulate.

To show this, consider the generalization (Belavkin, 1994) of the projection postulate to the continuous spectrum case. This states that after the measurement returning a result τ, the state of the system is given by the normalization $\psi_\tau = G(\tau)\, \psi/\|G(\tau)\psi\|$ of a linear transform $G(\tau)\psi$ of the *a priori* state vector ψ given by a family $\{G(\tau)\}$ of Hilbert space operators $G(\tau)$ with $\int_{-\infty}^{\infty} G(\tau)^\dagger G(\tau) d\tau = \hat{1}$. The operator-valued measure Π_Δ is then defined by the integration

$$\Pi_\Delta = \int_\Delta G^\dagger(\tau) G(\tau) d\tau$$

of the operator-valued density $\Pi(\tau) = G^\dagger(\tau)G(\tau)$ of this measure. However, because of the continuity of time τ there are no eigenprojectors corresponding to the continuous values $\tau \in \mathbb{R}$. Therefore, instead of the orthoprojectors, some other, nonorthogonal reduction operators $G(\tau)$ corresponding to an unsharp time measurement must be used to obtain a Hilbert space state vector $G(\tau)\psi$ with $\|G(\tau)\psi\| < \infty$ for (almost) each result τ of the measurement.

For a candidate measurement operator $G(\tau)$ to make physical sense, it must satisfy certain conditions. Two of these have already been dealt with, but the following still remain:

(i) The family $\{G(\tau)\}$ must commute with the energy coshift, $[G(\tau), V(\lambda)] = 0$, $\lambda \in \mathbb{R}_+$, so that the nondemolition measurement of time will be compatible with the ideal time measurement, described by the time vectors $|i\mathrm{HS}^{-1}\,\tau)$, i.e.,

$$(s|G(\tau) = g_s(\tau)(s|, \qquad \tau \in \mathbb{R}$$

where $g_s(\tau)$ are complex L^2 $(\mathbb{R})$-functions.

(ii) The family $\{G(\tau)\}$ must be covariant with respect to the time shift,

$$e^{-iHt/\hbar}G(\tau - t) = e^{i\theta(t)}G(\tau)e^{-iHt/\hbar}, \qquad t \in \mathbb{R}$$

where $\theta(t) \in (0, 2\pi)$, so that the predicted physics is unchanged by our choice of the origin for time.

One can easily show that the ideal unsharp measurement examined above is not compatible with these conditions, because there is no such covariant $G(\tau)$ that commutes with $V_\lambda^\dagger$ for which

$$\int_\Delta G(\tau)^\dagger G(\tau)\, d\tau = \Pi_\Delta = \frac{1}{2\pi\hbar}\int_\Delta |i\hbar^{-1}\tau)(i\hbar^{-1}\tau|\, d\tau$$

for any measurable $\Delta \subset \mathbb{R}$,

Suppose that this were so. We know that the probability density of τ is given by $|h(\tau)|^2$. On the other hand, from the commutativity with $G(\tau)$, it follows that the *a posteriori* state vector ψ_τ is obtained by modulation by some filter (or envelope) function $g_s(\tau)$ in the s-representation, and then by the normalization

$$\eta_\tau(s) = (s|\psi_\tau = \frac{g_s(\tau)\eta(s)}{c(\tau)}$$

where

$$|c(\tau)|^2 = \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=0} |g_s(\tau)|^2\|\eta(s)\|^2\, ds$$

is the corresponding probability density. As these two expressions for the probability density must be equal, and since $\|\eta\,(s)\|^2 = 2\pi\hbar\|h(\tau)\|^2$, where $s = i\hbar^{-1}\tau'$, then $|g_s(\tau)|^2$ must be a delta function. There is, however, no such square-integrable function $g_s(\tau)$.

3. A REALIZATION OF UNSHARP MEASUREMENT OF TIME

The covariant measurement operators $G(\tau)$ can be obtained from the interaction model with a clock, generalizing the model (Belavkin, 1994; Stratonovich and Belavkin, 1996) with the discrete spectrum. The non-Hermitian model for such an interaction is similar to the model for nondemolition measurements of quantum phase (Belavkin and Bendjaballah, 1994). It is given by the nonunitary interaction operator $V_{-\hat{x}} = P_0 \exp[i(\hat{\tau} \otimes \hat{x})/\hbar]$, defining $G(\tau) = P_0\varphi(\tau - \hat{\tau})$ as

$$G(\tau)\psi(\varepsilon) = (\langle\varepsilon| \otimes \langle\tau|)V_{-\hat{x}}(\psi \otimes \varphi) = \psi(\varepsilon - \hat{x})\varphi(\tau)$$

Here $\hat{x}$ is the momentum operator of the clock pointer; $|\tau\rangle$, $\tau \in \mathbb{R}$ are the generalized eigenvectors of the self-adjoint operator P: $f \mapsto i\hat{\hbar}f'$ in $L^2(\mathbb{R})$, describing the continuous pointer position $p = \tau$ in the momentum representation $\hat{x}f(x) = xf(x)$, and $\varphi(\tau) = \tilde{f}(\tau)$ is a clock wavefunction, given as the involute transform

$$\tilde{f}(\tau) = (2\pi\hbar)^{-1/2} \int_{-\infty}^{\infty} \bar{f}(x)e^{itx/\hbar}\, dx$$

of the admissible initial state $f(x) = 0$, $x > 0$, $\|\varphi\|^2 = \int_{-\infty}^{0} |f(x)|^2\, dx = 1$ (with negative pointer momentum.) Because the admissible wavefunctions cannot be localized in the position representation, the time measurement is always unsharp, but it can be made almost sharp by choosing $f(x) = 1/\sqrt{E}$ for $x \in [-E, 0]$ and $f(x) = 0$ for $x \notin [-E, 0]$ and going to the limit $E \to \infty$.

Consider as an example the cases where the continuous energy spectrum is in the range (a) $[0, \infty)$ and (b) $[0, E]$.

(a) Take the Hilbert space of the clock to be given by the linear span of $\{|x\rangle\, |x \in (-\infty, 0]\}$ as in the discrete case. Then consider a wavefunction of the clock in the momentum x-representation of the form $f(x) = (2\lambda)^{1/2} e^{\lambda x}$, with $\lambda > 0$ real. This is normalized and we can find the wavefunction of the pointer in the position representation, given by $\varphi(\tau) = \langle\tau|\varphi$, which is

$$\varphi(\tau) = \left(\frac{\hbar\lambda}{\pi}\right)^{1/2} \frac{1}{\hbar\lambda + i\tau}$$

Hence the probability distribution of p is $|\varphi(\tau)|^2 = \hbar\lambda/\pi\, (\tau^2 + \hbar^2\lambda^2)$.

Now, since $\langle\tau|V_{-\hat{x}}\varphi = P_0\varphi(\tau - \hat{\tau})$ and $\Pi(\tau) = G(\tau)^\dagger G(\tau)$ is the operator-valued density for time measure, then $|\varphi(\tau)|^2$ gives the probability of measuring a time different from the mean time by $p = \tau$. There are thus two possibilities for the sharp measurement of time:

(i) In the classical limit, $\hbar = 0$, we obtain $|\varphi(\tau)|^2$ as a delta function corresponding to the exact classical measurement of time.

(ii) In the limit as $\lambda \to 0$, we again find that $|\varphi(\tau)|^2$ takes the form of a delta function and hence get a sharp measurement of time. We cannot, however, take $\lambda = 0$, but since the height of the function $|\varphi(\tau)|^2$ at $\tau = 0$ is $1/(\hbar\lambda)$ we effectively obtain sharp measurement of time when $\lambda \leq 1/\hbar$.

(b) The clock Hilbert space is now $\{|x\rangle \; |x \in [-E, 0]\}$. Hence the normalized clock wavefunction in the momentum representation becomes

$$f(x) = \left(\frac{2\lambda}{1 - e^{-2\lambda E}}\right)^{1/2} e^{\lambda x}$$

So in the position representation we obtain

$$|\varphi(\tau)|^2 = \frac{\hbar\lambda}{\pi(1 - e^{-2\lambda E})(\tau^2 + \hbar^2\lambda^2)} \left(1 - 2e^{-\lambda E} \cos \frac{\tau E}{\hbar} + e^{-2\lambda E}\right)$$

As before, if we consider the classical limit, $\hbar = 0$, then we obtain the sharp measurement of time. Now consider the limit $\lambda \to 0$. In this case, we find that, when $\tau \neq 0$,

$$|\varphi(\tau)|^2 \to \frac{\hbar[1 - \cos(\tau E/\hbar)]}{\pi E \tau^2}$$

and when $\tau \to 0$, $|\varphi(\tau)|^2 \to E/(2\pi\hbar)$. Hence it is not sufficient to have $\lambda = 0$ (which is equivalent to the clock wavefunction $f(x) = 1/\sqrt{E}$ for $x \in [-E, 0]$ and $f(x) = 0$ for $x \notin [-E, 0]$) but we must also take $E \to \infty$, as noted above, in order to obtain a sharp measurement.

REFERENCES

Belavkin, V. P. (1994). Nondemolition principle of quantum measurement theory, *Foundations of Physics,* **24**, 685.

Belavkin, V. P., and Bendjaballah, C. (1994). Continuous measurements of quantum phase, *Quantum Optics,* **6**, 169–186.

Helstrom, C. W. (1976). *Quantum Detection and Estimation Theory*, Academic Press, New York.

Holevo, A. S. (1978). *Reports on Mathematical Physics,* **13**(3), 287–307.

Holevo, A. S. (1982). *Probabilistic and Statistical Aspects of Quantum Theory*, North-Holland, Amsterdam.

Stratonovich, R. L., and Belavkin, V. P. (1996). On the dynamical interpretation of the quantum measurement postulate, **35**(11), 2215–2228.

Von Neumann, J. (1955). *Mathematical Foundation of Quantum Mechanics*, Princeton University Press, Princeton, New Jersey.